(Homogeneous Linear ODE with constant coefficients)

Homogeneous
+
Linear
+
Constant coefficients.

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots a_{n-1} y' + a_n y = 0,$$

where a_i's are constant $\forall i = 0, 1, \cdots n$

Let us think of a function y which has derivatives proportional to y.

$$y = e^{mx}$$

$$y = e^{mx}$$

$$y' = m e^{mx}, \quad y'' = m^2 e^{mx}$$

n^{th} order linear ODE (possibly variable coefficient) then for $n=2$ Guess 1st solution (say u) Then second solution $y = uv$ can be obtained by order reduction method. (Homogeneous)

$y = me^{mx}$, $y = me^{1}$

$y^{(n)} = m^n e^{mx}$

$$a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots a_n e^{mx} = 0$$

$$\left(a_0 m^n + a_1 m^{n-1} + \cdots a_{n-1} m + a_n\right) e^{mx} = 0$$

$$\Rightarrow \boxed{a_0 m^n + a_1 m^{n-1} + \cdots a_{n-1} m + a_n = 0}$$

Auxiliary Equation

characteristic Equation.

Ex:

$$y'' - 3y' + 2y = 0$$

Auxiliary Equation

$$m^2 - 3m + 2 = 0$$

$$m = 2, 1$$

$$y_1 = e^{2x}, \quad y_2 = e^{x}$$

$$y(x) = c_1 e^{2x} + c_2 e^{x}$$

general solution

Case 1 : When all roots of auxiliary equation are distinct and real say $m_1, m_2, \ldots m_n$, then $e^{m_1 x}, e^{m_2 x} \ldots e^{m_n x}$ are linearly independent solutions. A general solution, in this case, can be written as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots c_n e^{m_n x}.$$

Ex. $y'' - 6y' + 9y = 0$

A.E. $m^2 - 6m + 9 = 0$

$m = 3, 3$

Thus e^{3x} is a solution.

$$y = v(x)\, e^{3x}$$

Find $v(x)$ using order reduction meth

$v(x) = Ax + B$

e^{3x}, xe^{3x} are two linearly independent solution

$y(x) = c_1 e^{3x} + c_2 x e^{3x}$

$y(x) = (c_1 + c_2 x) e^{3x}$ — general solution.

Case 2 : When roots of Auxiliary equation are repeated.

$m_1, m_2, \cdots m_n$

$m = m_1 \rightarrow$ (k_1) times repeated

$e^{mx}, xe^{mx}, x^2e^{mx} \cdots x^{k_1-1}e^{mx}, e^{m_2x} \cdots e^{m_nx}$

Thus general solution

$$y(x) = (c_1 + c_2x + c_3x^2 + \cdots c_{k_1}x^{k_1-1})e^{m_1x} + c_{k_1+1}e^{m_2x} + \cdots c_ne^{m_nx}$$

Ex: Assume 2, 2, 2, 3, 3, 4 are roots of auxiliary equation of an ODE. General solution?

$$(c_1 + c_2x + c_3x^2)e^{2x} + (c_4 + c_5x)e^{3x} + c_6e^{4x}$$

$(c_1 + c_2 x + c_3 x)e + (c_4 + c_5 x)c + c_6$

Ex: $y'' + y = 0$ $\qquad ay'' + by' + cy = 0$

$m^2 + 1 = 0$.

$m = \pm i$

$y = f + ig$

$(af'' + bf' + cf) + i(ag'' + bg' + cg) = 0$

e^{ix} $\qquad e^{-ix}$

$y(x) = c_1 e^{ix} + c_2 e^{-ix}$

$= c_1(\cos x + i \sin x) + c_2(\cos x - i \sin x)$

$= (c_1 + c_2)\cos x + i(c_1 - c_2)\sin x$

$y(x) = c_1 \cos x + c_2 \sin x.$

Case 3: When roots are complex.

$m_1 = a + ib, \; m_2 = a - ib$

$(a+ib)x \qquad (a-ib)x$

e^{ax}, e^{ax}

$e^{ax}e^{ibx}, e^{ax}.e^{-ibx}$

$e^{ax}(\cos bx + i\sin bx),\ e^{ax}(\cos bx - i\sin bx$

$y(x) = C_1 e^{ax}(\cos bx + i\sin bx) + C_2 e^{ax}(\cos bx - i\sin bx$

$= e^{ax}(C_1+C_2)\cos bx + e^{ax}i(C_1-C_2)\sin bx$

$$\boxed{y(x) = e^{ax}(C_1\cos bx + C_2\sin bx)}$$

Ex: $y^{iv}(x) - 4y'''(x) + 14y'' - 20y' + 25y = 0$

$m^4 - 4m^3 + 14m^2 - 20m + 25 = 0$

$m = 1+2i,\ 1+2i$

$$e^x(c_1\cos 2x + c_2\sin 2x) + xe^x(c_3\cos 2x + c_4\sin 2x)$$

$$y(x) = e^x\left[(c_1+c_3x)\cos 2x + (c_2+xc_4)\sin 2x\right]$$

Higher Order Linear ODE

Theorem 1 Let $a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} \cdots a_{n-1}y' + a_n y = f(x)$, $a \le x \le b$

& $y(x_0) = c_1$, $y'(x_0) = c_2 \cdots y^{n-1}(x_0) = c_n$

for some $x_0 \in [a, b]$ & c_i constants $\forall i$

Then the ODE has unique solution $a \le x \le b$

provided function

$a_i(x)$ & $f(x)$ are continuous $\forall i = 0, \cdots n$

& $x \in [a, b]$

$$(*) \begin{cases} a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \quad \forall I \subset \mathbb{R} \\ y(x_0) = 0, \quad y'(x_0) = 0 \\ x_0 \in I \end{cases}$$

In light of above theorem ODE in (*) has only trivial solution.

$\underline{n^{th}}$ order $\underline{\text{Linear}}$ ODE has n-linearly independent solutions.

If $a_i(x)$ are continuous then

2) $a_0 y^{(n)}(x) + \cdots \; a_{n-1} y' + a_n y = 0$ has n-linearly independent solutions.

Let $y_1, y_2, \ldots y_n$ be n-linearly independent solutions of 2) then

$y = \sum_{i=1}^{n} c_i y_i$ is called a general solution of 2) where

c_i's are constant $\forall i = 1, 2, \ldots n$.

Moreover any solution $\hat{\phi}$ of 2) will be of the form

$$\phi(x) = \sum_{i=1}^{n} d_i y_i \text{ for some}$$

specific (depending upon ϕ)

constants d_i $\forall i = 1, 2, \ldots n$.

$$S = \{ y \in C^n(I; \mathbb{R}) : y \text{ solves 2)} \}$$

Then S is a vector space under usual vector addition & scalar multiplic of V. space of functions.

$$(y_1 + y_2)(x) = y_1(x) + y_2(x) \quad \forall x \in I$$

$$(cy)(x) = c\, y(x) \quad \forall c \in \mathbb{F}, \quad \forall x \in I.$$

$\dim S = n$

$$L(y) \equiv a_0(x)y'' + a_1(x)y' + a_2(x)y = f(x) \quad \text{---(1)}$$

Homogeneous ODE

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \quad \text{(HLDE)}$$

Assume that y_1 and y_2 solves HLDE and y_1, y_2 are L.I.

Let ϕ be any solution of (1).

Then $y = c_1y_1 + c_2y_2 + \phi$ is a general solution of 1)

$$L(y) = c_1L(y_1) + c_2L(y_2) + L(\phi)$$
$$= c_1 0 + c_2 0 + f(x)$$
$$= f(x)$$

1. How to find out solutions (Linearly independent) of HLDE

2. How to guess such ϕ.

Reduction of Order

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

Observe that $y = x$ is a solution.

Let $y = xv(x)$ be a solution of given ODE, then

$$y' = v + xv'$$
$$y'' = v' + xv'' + v'$$

$$(x^2+1)[v' + xv'' + v'] - 2x[v + xv'] + 2xv = 0$$

$$(x^2+1)xv'' + 2(x^2+1)v' - 2xv - 2x^2v' + 2xv = 0$$

$$(x^2+1)xv'' + [2x^2 + 2 - 2x^2]v' = 0$$

$$(x^2+1)xv'' + 2v' = 0$$

$$v' = w$$

$$(x^2+1)\,x\,w' + 2w = 0$$

(variable separable)

$$\frac{dw}{w} = -\frac{2}{x(x^2+1)}\,dx.$$

$$\frac{dv}{dx} = w = -\int \frac{2}{x(x^2+1)}\,dx = c\,\frac{(x^2+1)}{x^2}$$

$$v = c\int \frac{x^2+1}{x^2}\,dx$$

$x - \frac{1}{2x}$

$y = x \cdot v$ a solution.

$$y(x) = c_1 x + c_2 x v$$

Example: $(2x+1)\,y'' - 4(x+1)\,y' + 4y = 0$

$2x$

$y_1(x) = e$

$y' = 2e^{2x} v(x) + e^{2x} v'$

$y'' = 4e^{2x} v + 2e^{2x} v' + e^{2x} v''$
$+ 2e^{2x} v'$

$(2x+1)\,4 - 4(x+1)\,2 + 4$

$2x+1 - 2(x+1) + 1 = 0$

Assume that $y(x) = e^{2x} \cdot v(x)$ solves given ODE

$(2x+1)\left[4e^{2x} v + 4e^{2x} v' + e^{2x} v''\right]$

$- 4(x+1)\left[2e^{2x} v + e^{2x} v'\right]$

$+ 4ve^{2x} = 0$

$(2x+1)\left[4v + 4v' + v''\right]$

$- 4(x+1)\left[2v + v'\right] + 4v = 0$

$(2x+1)v'' + v'\left[4(2x+1) - 4(x+1)\right]$

$\left[4(2x+1) - 8(x+1) + 4\right] v = 0$

$8x + 4 - 8x - 8 + 4$

$= 0$

$v' = w$

$$(2x+1)w' + w(4x) = 0$$

$$\int \frac{dw}{w} = -\int \frac{4x}{2x+1}\,dx$$

$$\frac{dv}{dx} = w = -\int \frac{4x}{2x+1}\,dx$$

$$v = -\left(\left(\int \frac{4x}{2x+1}\,dx\right)dx + c\right)$$

Non-Homogeneous Linear Equations with constant coefficient

We have seen

(*) $a_0 y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots a_{n-1} y'(x) + a_n y = f(x)$

when $f(x) \equiv 0$

a_i's are constant

$y = e^{mx}$ solves (*)

a_i may not be constant

$n = 2$

Order Reduction method

1) If roots of Auxiliary equation are distinct and real

say $m_1, m_2, \cdots m_n$

then $e^{m_1 x}, e^{m_2 x}, \cdots e^{m_n x}$ are n - L.I. solutions

2) when roots of A.E. are repeated (real)

say $m_1, m_2, \cdots m_n$, where

$m_1 = m_2 = \cdots m_k = m$, $m_{k+1} \cdots m_n$ are

distinct, then

$$e^{m_1 x}, xe^{m_1 x}, \ldots x^{k-1}e^{m_1 x}, e^{m_{k+1} x}, \ldots e^{m_n x}$$

are n- L.I. solutions.

3) when roots are complex (degree $n=2$)

say $a \pm ib$

then $e^{ax}\cos bx$, $e^{ax}\sin bx$ are linearly independent solution.

When $f(x) \neq 0$

Ex. $y'' - 2y' - 3y = 2e^{4x}$

Say $y = Ae^{4x}$ solves it, where A is constant

$$\therefore 16Ae^{4x} - 8Ae^{4x} - 3Ae^{4x} = 2e^{4x}$$

$$(16A - 8A - 3A - 2)e^{4x} = 0$$

$$(5A - 2)e^{4x} = 0$$

$$\Rightarrow A = 2/5$$

general soln for HLDE $\Rightarrow y = \frac{2}{5}e^{4x}$ is a solution

$y = y_c + y_p \rightarrow$ Non homogeneous ODE

$\hookrightarrow$ Any solution of NHLDE

$$m^2 - 2m - 3 = 0$$

$$m = 3, -1$$

$$y(x) = c_1 e^{3x} + c_2 e^{-x} + \frac{2}{5}e^{4x}$$

general solution of given NHLDE.

Ex: $y'' - 2y' - 3y = 2e^{3x}$

Let $y(x) = Ae^{}$ → Does not satisfying given eqn for any A.

e^{3x}, e^{-x} are L.I. solution of HLDE.

Let $y(x) = Axe^{3x}$ solves given DE.

$\Rightarrow A = 1/2$

Therefore, $y(x) = c_1 e^{3x} + c_2 e^{-x} + \frac{1}{2}xe^{3x}$

$f(x) = x^n, e^{ax}, \sin(bx+c), \cos(bx+c)$

Method of undetermined coefficients

	UC function	UC set

	x^n	$\{1, x, x^2, \dots x^n\}$
	e^{qx}	$\{e^{qx}\}$
	$\sin(bx+c)$ or $\cos(bx+c)$	$\{\sin(bx+c), \cos(bx+c)\}$
	$x^n e^{qx}$	$\{e^{qx}, xe^{qx}, x^2e^x, \dots x^n e^{qx}\}$
	$x^n \sin(bx+c)$ or $x^n \cos(bx+c)$	$\{\cos(bx+c), \sin(bx+c), x\cos(bx+c), x\sin(bx+c), x^2\cos(bx+c), x^2\sin(bx+c) \dots x^n\cos(bx+c), x^n\sin(bx+c)\}$
	$e^{qx}\cos(bx+c)/\sin(bx+c)$	$\{e^{qx}\cos(bx+c), e^{qx}\sin(bx+c)\}$

Ex: $y'' - 3y' + 2y = 4x^2$

AE: $m^2 - 3m + 2 = 0$

$m = 1; 2$

$y_c(x) = C_1 e^x + C_2 e^{2x}$

$y_p(x) = A \cdot 1 + B \cdot x + C \cdot x^2$

y_p

$$y_p'(x) = B + 2Cx$$

$$y_p''(x) = 2C$$

$$2C - 3(B + 2Cx) + 2(A + Bx + Cx^2) = 4x^2$$

$$(2C - 4)x^2 + (2B - 6C)x + (2C - 3B + 2A) = 0$$

$\forall x \in I$

$$\Rightarrow \quad 2C - 4 = 0 \Rightarrow C = 2$$

$$2B - 6C = 0 \Rightarrow B = 6$$

$$2C - 3B + 2A = 0 \Rightarrow A = 7$$

4 −18

$$y_p(x) = 7 + 6x + 2x^2$$

$$y(x) = C_1 e^{x} + C_2 e^{2x} + 2x^2 + 6x + 7$$

Example :

$$y'' - 3y' + 2y = x^2 e^x$$

$$y_c(x) = c_1 e^x + c_2 e^{2x}$$

UC function : $x^2 e^x$

UC set $S = \{e^x, xe^x, x^2 e^x\}$

$S' = \{xe^x, x^2 e^x, x^3 e^x\}$ as e^x is a solution of homogeneous L.D.E.

$$y_p(x) = Axe^x + Bx^2 e^x + Cx^3 e^x$$

Ex: $y'' - 2y' + 1 = x e^x$

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1$$

$$y_c(x) = C_1 e^x + C_2 x e^x$$

uc function: $x^2 e^x$

uc set $S = \{e^x, x e^x, x^2 e^x\}$

$$S' = \{x e^x, x^2 e^x, x^3 e^x\}$$

$$S'' = \{x^2 e^x, x^3 e^x, x^4 e^x\} \checkmark$$

$$y_p(x) = A x^2 e^x + B x^3 e^x + C x^4 e^x$$

Nonhomogeneous linear differential equations with constant coefficients

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$$a_0 y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots \quad a_{n-1} y'(x) + a_n y = f(x)$$

where a_i's are constant.

$$f(x) = x^n, \; e^{ax}, \; \sin(bx+c), \; \cos(bx+c)$$

UC function	UC set
x^n	$\{1, x, x^2, \ldots x^n\}$
e^{ax}	$\{e^{ax}\}$
$\sin(bx+c)$ or $\cos(bx+c)$	$\{\sin(bx+c), \cos(bx+c)\}$
$x^n e^{ax}$	$\{e^{ax}, x e^{ax}, x^2 e^{ax} \ldots x^n e^{ax}\}$
$x^n \cos(bx+c)$ or $x^n \sin(bx+c)$	$\{\cos(bx+c), \sin(bx+c), x\cos(bx+c), x\sin(bx+c), x^2\cos(bx+c), x^2\sin(bx+c)$

$x^n \sin(bx+c)$	$\cdots \; x^n\cos(bx+c),\ x^n\sin(bx+c)\}$
$e^{ax}\sin(bx+c)$ or $e^{ax}\cos(bx+c)$	$\{e^{ax}\cos(bx+c),\ e^{ax}\sin(bx+c)\}$

$x^n e^{ax}$

$\{1, x, x^2, \cdots x^n\}$ $\{e^{ax}\}$

$x^n \cos(bx+c)$

$\{1, x, x^2, \cdots x^n\}$ $\{\sin(bx+c), \cos(bx+c)\}$

Example $y'' - 2y' - 3y = 2e^x - 10\sin x$

$y_c \rightarrow$ complementary function

AE: $\Rightarrow m^2 - 2m - 3 = 0$

$m = 3, -1$

$y(0) = 2$

$y'(0) = 4$

$y_c(x) = c_1 e^{-x} + c_2 e^{3x}$

$y_c(x) = \ldots$

$e^x, \quad \sin x$

$S_1 = \{e^x\}, \; S_2 = \{\sin x, \cos x\}$

$$y_p(x) = Ae^x + B\sin x + C\cos x$$

On solving

$$A = -\tfrac{1}{2} \quad B = 2, \quad C = -1$$

General solution

$$y(x) = y_c(x) + y_p(x)$$

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{3x} - \frac{1}{2}e^x + 2\sin x - \cos x}$$

$y(0) = 2, \Rightarrow c_1 + c_2 - \tfrac{1}{2} - 1 = 2 \Rightarrow c_1 + c_2 = \tfrac{7}{2}$

$y'(0) = 4 \Rightarrow -c_1 + 3c_2 - \tfrac{1}{2} + 2 = 4 \Rightarrow -c_1 + 3c_2 = \tfrac{5}{2}$

Example

$y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$

y_c – complementary function

$4C_2 = 6$

$C_2 = 3/2$

$C_1 = 1$

A.E.: $m^2 - 3m + 2 = 0$

$m = 2, 1$

$y_c(x) = c_1 e^x + c_2 e^{2x}$

$x^2 \qquad e^x \qquad xe^x \qquad e^{3x}$

$S_1 = \{1, x, x^2\} \quad S_2 = \{e^x\} \quad S_3 = \{e^x, xe^x\} \quad S_4 = \{e^{3x}\}$

$S_2 \subset S_3$

Ignore S_2

$S_1, \; S_3, \; S_4$

Revised UC set: $S_1, \; S_3' = \{xe^x, x^2e^x\}, \; S_4$

$y_p(x) = A + Bx + Cx^2 + Dxe^x + Ex^2e^x$

$$+Fe^{3x}$$

$$y_p'(x) = B + 2Cx + De^x + Dxe^x + 2Exe^x + Ex^2e^x + 3Fe^{3x}$$

$$= B + 2Cx + De^x + (D+2E)xe^x + Ex^2e^x + 3Fe^{3x}$$

$$y_p''(x) = 2C + De^x + De^x + Dxe^x + 2Ee^x + 2Exe^x + 2Exe^x + Ex^2e^x + 9Fe^{3x}$$

$$y_p''(x) = 2C + (2D+2E)e^x + (D+4E)xe^x + Ex^2e^x + 9Fe^{3x}$$

$$2C + (2D+2E)e^x + (D+4E)xe^x + Ex^2e^x + 9Fe^{3x} - 3B - 6Cx - 3De^x - 3(D+2E)xe^x - 3Ex^2e^x - 9Fe^{3x} + 2A + 2Bx + 2Cx^2 + 2Dxe^x + 2Ex^2e^x + 2Fe^{3x} = f(x)$$

$$\checkmark\ 2C - 3B + 2A = 0$$

$$2D + 2E - 3D = 2E - D = 1 \Rightarrow D = 2E - 1$$

$$\boxed{D = -3}$$

$$2C = 2 \cdot \Rightarrow \boxed{C = 1}$$

$$D + 4E - 3(D + 2E) + 2D = 2$$

$$-2E = 2 \Rightarrow \boxed{E = -1}$$

$\leftarrow$ $E - 3E + 2E = 0$

$$9F - 9F + 2F = 4 \Rightarrow \boxed{F = 2}$$

$$-6C + 2B = 0$$

$$B = 3C \Rightarrow \boxed{B = 3}$$

$$2A = 3 \times 3 - 2 \times 1 = 7$$

$$A = 7/2, \; B = 3, \; C = 1, \; D = -3$$

$$E = -1, \; F = 2$$

$$\boxed{y_p(x) = 7/2 + 3x + x^2 - 3xe^x - x^2 e^x + 2e^{3x}}$$

Ex :- $\frac{d^4 y}{dx^4} + \frac{d^2 y}{dx^2} = x^2$

Ex: $\frac{d^4y}{dx^4} + \frac{dy}{dx^2} = 3x + 4\sin x - 2\cos x$

$m^4 + m^2 = 0$

$m = 0, 0, \pm i$ (0, 0)

$y_c(x) = c_1 + c_2 x + c_3 \sin x + c_4 \cos x$

(c_1 ↑ x^2; $c_2 x$ ↑ $\sin x$; $\cos x$)

UC set: $S_1 = \{1, x, x^2\}$ $S_2 = \{\sin x, \cos x\}$ $S_3 = \{\sin x, \cos x\}$

$S_2 = S_3$

Revised UC set: S_1, S_2

$S_1' = \{x, x^2 x^3\}$, $S_2' = \{x\sin x, x\cos x\}$ (↑ x)

$S_1'' = \{x^2, x^3, x^4\}$, $S_2' = \{x\sin x, x\cos x\}$

$y_p(x) = Ax^2 + Bx^3 + Cx^4 + Dx\sin x + Ex\cos x$

y_p

On solving

$$y_p(x) = -3x^2 + \frac{1}{4}x^4 + x\sin x + 2x\cos x$$

$A=-3,\ B=0,\ C=\frac{1}{4},\ D=1,\ E=2$

Method of Variation of parameters

So far, we have seen

$$a_0 y^{(n)}(x) + a_1 y^{n-1}(x) \cdots \quad a_{n-1} y' + a_n y = f(x),$$

where a_i's are constant and

$f(x) = x^n, e^{ax}, \sin x, \cos x$

Let us restrict ourselves to $n = 2$

$$a_0(x) y'' + a_1(x) y' + a_0(x) y = f(x) \quad — (*)$$

Let y_1, y_2 are two linearly independent solutions of corresponding Homogeneous eq^n

i.e. $a_0(x) y'' + a_1(x) y' + a_0(x) y = 0$

Let $y_p(x) = v_1(x) y_1(x) + v_2(x) y_2(x)$ solves $(*)$ -(1)

$$y_p'(x) = v_1(x) y_1'(x) + v_1'(x) y_1(x) + v_2(x) y_2'(x)$$

$$+ v_2'(x) y_2(x)$$

Without violating our assumption in (1), assume that

$$v_1'(x) y_1(x) + v_2'(x) y_2(x) = 0 \quad \text{—} (2)$$

$$y_p''(x) = v_1(x) y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'$$

$$a_0 [v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'] + a_1 [v_1 y_1' + v_2 y_2']$$

$$+ a_2 [v_1 y_1 + v_2 y_2] = f(x)$$

$$v_1 [a_0 y_1'' + a_1 y_1' + a_2 y_1] + v_2 [a_0 y_2'' + a_1 y_2' + a_2 y_2]$$

$$+ a_0 v_1' y_1' + a_0 v_2' y_2' = f(x)$$

$$a_0 v_1' y_1' + a_0 v_2' y_2' = f(x) \quad \text{—} (3)$$

From (2) & (3)

Eqn (2) - (3)

$$\begin{bmatrix} y_1 & y_2 \\ a_0 y_1' & a_0 y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

On solving,

$$v_1'(x) = \frac{-f(x)\, y_2(x)}{a_0(x)\, W(y_1, y_2)}$$

$$v_2'(x) = \frac{f(x)\, y_1(x)}{a_0(x)\, W(y_1(x), y_2(x))}$$

Thus we can obtain $v_1(x)$ & $v_2(x)$ upon integration.

The general solution $\boxed{y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)}$

Ex: $y'' + y = \tan x$

$\sin x$ & $\cos x$ are two linearly independent

Solutions of homogeneous equation

$(m^2+1=0 \Rightarrow m = \pm i)$.

$$y_p(x) = v_1(x)\sin x + v_2(x)\cos x$$

$$v_1' \sin x + v_2' \cos x = 0 \quad \checkmark$$

$$W(\sin x, \cos x) = -1$$

$$v_1' = \sin x$$

$$\Rightarrow v_1(x) = -\cos x + a$$

$$v_2' = \cos x - \sec x$$

$$v_2(x) = \sin x - \log|\sec x + \tan x| + b$$

$$y_p(x) = (-\cos x + a)\sin x + (\sin x - \ln|\sec x + \tan x| + b)\cos x$$

$$= a\sin x + b\cos x - \cos x(\log|\sec x + \tan x|)$$

$$y(x) = c_1 \sin x + c_2 \cos x + y_p(x)$$

$$\boxed{y(x) = A\sin x + B\cos x - \cos x(\log|\sec x + \tan x|)}$$

Cauchy- Euler DE

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots \quad a_{n-1} x \frac{dy}{dx} + a_n y = f(x)$$

a_i's are constant.

$$\text{Let } x = e^t \Rightarrow t = \log x \qquad dt/dx = \frac{1}{x}$$

$$a_0 x^2 y'' + a_1 x y' + a_2 y = f(x)$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} \Rightarrow \boxed{x\frac{dy}{dx} = \frac{dy}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\left(\frac{d^2y}{dt^2}\frac{dt}{dx}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

$$\Rightarrow \boxed{x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}}$$

$$a_0\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + a_1\frac{dy}{dt} + y = f(e^t)$$

$$a_0\frac{d^2y}{dt^2} + (a_1 - a_0)\frac{dy}{dt} + y = f(e^t)$$

Which has constant coefficients.

Example:

$$x^2y'' - 2xy' + 2y = x^3$$

Let $x = e^t$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2y}{dt^2} - \frac{1}{x}\frac{dy}{dt}$$

$$dx^2 \quad x^2\,dt^2 \quad x^2\,dt$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2\frac{dy}{dt} + 2y = e^{3t}$$

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$$

AE: $m^2 - 3m + 2 = 0$

$\Rightarrow m = 2, 1$

$$y_c(t) = c_1 e^{t} + c_2 e^{2t}$$

$$y_p(t) = \frac{1}{2}e^{3t} \quad (\text{using UC meth}$$

$$y(t) = c_1 e^{t} + c_2 e^{2t} + \frac{1}{2}e^{3t}$$

$$\boxed{y(x) = c_1 x + c_2 x^2 + \frac{1}{2}x^3}$$

Laplace Transform

$$F(s) := \mathcal{L}f(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Well defined: If f is bounded or piecewise continuous

$\mathcal{L}\{f\}$ ← Laplace transform of f

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt = \frac{e^{-st}}{-s}\Big|_0^\infty = \frac{1}{s}(1) = \frac{1}{s}$$

$s > 0$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t\, dt = t\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, dt$$

$$= -\frac{1}{s}(0) + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}$$

$s > 0$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt = \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty$$

$$= \frac{1}{a-s}\left[-1\right] = \frac{1}{s-a}$$

- $\mathcal{L}\{\sin bt\} = \dfrac{b}{s^2+b^2}, \quad s>0$

Properties of Laplace transform

* It is linear.

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}$$

* Let $\mathcal{L}\{f'(t)\}$ exists then

* $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$

* Let $L\{f^{(n)}(t)\}$ exists then

$$\Rightarrow L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f'''(0) \cdots - f^{(n-1)}(0)$$

For $n=2$

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$$

* Translation: If $L\{f\}$ exists for $s>a$

$$L\{e^{at} f(s)\} = F(s-a)$$

where $F(s) = L\{f(t)\}$

* If $L\{f(t)\} = F(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Inverse Laplace Transform

Let F be a given function.

$L^{-1}\{F(s)\} = \tilde{f}(t)$, where

$$L\{f(t)\} = F(s)$$

Ex: Find $L^{-1}\left\{\frac{1}{s^2+6s+13}\right\}$

$$F(s) = \frac{1}{s^2+6s+13} = \frac{1}{(s+3)^2+4} = \frac{1}{2}\cdot\frac{2}{(s+3)^2+2^2}$$

$$F(s) = \frac{1}{2} L\{e^{-3t}\sin 2t\}$$

$$\boxed{L^{-1}\left\{\frac{1}{s^2+6s+13}\right\} = \frac{1}{2}e^{-3t}\sin 2t}$$

$$L^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$$

$$F(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L^{-1}F(s) = L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= 1 - \cos t$$

$$\boxed{L(\cos bt) = \frac{s}{s^2+b^2}}$$

Ex: $\frac{dy}{dt} - 2y = e^{5t}$

dt

$$y(0) = 3$$

$$y' - 2y = e^{5t}, \quad y(0) = 3$$

Take Laplace transform on both sides.

$$L\{y'(t)\} - 2L\{y(t)\} = L\{e^{5t}\}$$

$s > 5$ $\quad sL\{y(t)\} - y(0) - 2L\{y(t)\} = \frac{1}{s-5}$

Denote $L\{y(t)\} = Y(s)$

$$(s-2)Y(s) = \frac{1}{s-5} + 3$$

$$Y(s) = \frac{3s-14}{(s-5)(s-2)}$$

$$y(t) = L^{-1}\{Y(s)\} = \frac{8}{3}e^{-1} + \frac{1}{3}e^{5t}$$

$$\frac{3s-14}{(s-5)(s-2)} = \frac{8}{3}\frac{1}{s-2} + \frac{1}{3(s-5)}$$

Ex: $y'' - 2y' - 8y = 0$

$$y(0) = 3, \quad y'(0) = 6.$$

$$L\{y''(t)\} - 2L\{y'(t)\} - 8L\{y(t)\} = L\{0\} = 0.$$

$$s^2Y(s) - s\,y(0) - y'(0) - 2(sY(s) - y(0)) - 8Y(s) = 0$$

$$Y(s)(s^2 - 2s - 8) = 3s + 6 - 6 = 3s$$

$$Y(s) = \frac{3s}{\quad} = \frac{2}{s-4} + \frac{1}{s+2}$$

$$s^2 - 2s - 8$$

$$y(t) = 2e^{4t} + e^{-2t}$$

Ex: $y'' + y = e^{-2t}\sin t$

$$y(0) = 0, \quad y'(0) = 0$$

$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{1}{(s+2)^2+1}$$

$$Y(s) = \frac{1}{(s^2+1)\left((s+2)^2+1\right)}$$

$$= -\frac{1}{8}\,\frac{s}{s^2+1} + \frac{1}{8}\,\frac{1}{s^2+1} + \frac{1}{8}\,\frac{s+2}{(s+2)^2+1}$$

$$+ \frac{1}{8}\,\frac{1}{(s+2)^2+1}$$

$$y(t) = -\frac{1}{8}\cos t + \frac{1}{8}\sin t + \frac{1}{8}e^{-2t}\cos t + \frac{1}{8}e^{-2t}\sin t$$

$$= -\frac{1}{8}\cos t + \frac{1}{8}\sin t + \frac{1}{8}e^{-t}\cos t$$

$$+ \frac{1}{8}e^{-2t}\sin t$$

Try Exercise on pp 436-437 S.L. Ross.
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